Solvable multi-fingered hands for exact kinematic synthesis

Abhijit Makhal and Alba Perez-Gracia

Abstract Multi-fingered hands are kinematic chains with a tree topology, that is, with a set of common joints that span several branches and end-effectors. When performing dimensional kinematic synthesis with simultaneous tasks for all the end-effectors, a new solvability criterion needs to be applied that includes checking the solvability of sub-chains. This criterion yields as a result that not all possible topologies are solvable for a common number of positions for all end-effectors. This article shows and proves the solvability criterion and derives some properties of the kinematic chains with tree topology for a single branching and identical fingers.

Key words: kinematic synthesis, multi-fingered hands.

1 Introduction

Kinematic chains with a tree topology consist of several common joints that branch to a number of serial chains, each of them corresponding to a different end-effector. A typical example of a kinematic chain with a tree topology is a wristed, multi-fingered hand.

Compared to other topologies, the tree topologies have not been so widely studied. Kinematic analysis for applications in modular robots and robotic hands can be found in [8], [9], and [1], and dynamic analysis is found in [3] and [2]. Structural synthesis for multiple fingers with no wrist, considering grasping and manipulation requirements, are found in [4]. The first reference to kinematic design of tree topologies is found in [5].

The kinematic synthesis of these topologies presents particular challenges that are different of those that appear in single serial chains or in closed-loop systems. In

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particular, the kinematic synthesis of multi-fingered hands has been explored also in [7] and more extensively in [6].

When dealing with exact kinematic synthesis, one of the first steps is to calculate the maximum number of positions that can be used, which define the workspace of the chain. In the case of tree topologies, consider a task having the same number of positions for each of the multiple end-effectors; this means that we are dealing with a coordinated action of all those end-effectors, denoted as a *simultaneous task*.

In this paper we focus on the particular issues that appear in tree topologies when dealing with exact synthesis for simultaneous positions of all end-effectors. The solvability for the simultaneous task case presented in [6] is proved and developed in further detail, and a basic classification, together with some results regarding solvable multi-fingered chains with identical fingers, are included.

2 Tree Topologies

We denote a tree topology for a kinematic chain as that of a chain having a set of common joints spanning several chains and ending in multiple end-effectors. The tree topology is modeled using graph theory; for this we follow the approach of Tsai [10]. The kinematic chain is represented as a rooted graph, with the root vertex being fixed with respect to a reference system.

A tree topology is denoted as $SerialChain - (Branch_1, Branch_2, \ldots, Branch_b)$, where $SerialChain$ are the common joints and the dash indicates a branching, with the branches contained in the parenthesis, each branch $Branch_i$ characterized by its type and number of joints. In the case of using just revolute joints, the joint type is dropped and only the number of joints is indicated. Figure 1 presents the compacted graph for a $3R - (2R, R - (R, R, R))$, or $3 - (2, 1 - (1, 1, 1))$ chain, with two branches, one of them branching again on three additional branches, for a total of four end-effectors. Branches are ordered according to their branching order (branch 1 is the first one to branch). The root vertex is indicated with a double circle.

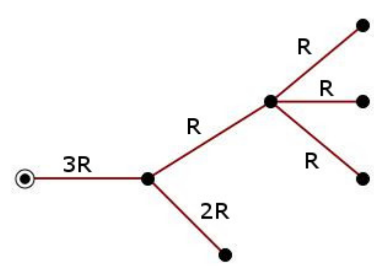


Fig. 1 A $3 - (2, 1 - (1, 1, 1))$ tree topology.

In tree topologies, a vertex can be connected to several edges defining several branches; a tree topology will always have links that are ternary or above, which are identified in the graph as a vertex spanning several edges. For the purpose of this work we consider serial chains in each branch, with only revolute joints. For the reduction of closed-loop chains to serial chains for synthesis purposes, see [6].

The contracted graph does not have binary vertices, and the primary vertices are either the root node or end-effectors. Having several end-effectors allows us to change the root node to any of them: consider displacements $\mathbf{P}_i$ to each end-effector i, where $i = 1$ is the root node. The inversion of the root from $i = 1$ to $i = j$ is

$$\mathbf{P}_i^* = \mathbf{P}_j^{-1}\mathbf{P}_i, \tag{1}$$

where $\mathbf{P}_i^*$ would be the i^{th} end-effector's position with respect to the new root node j.

3 Dimensional Kinematic Synthesis for Tree Topologies

Dimensional kinematic synthesis seeks to find the position of the joint axes for a given topology, in order for each of the end-effectors to perform a given set of displacements. In this section we present a summary of the design methodology when the chosen topology is that of multi-fingered hands; for details, see [6].

Given a set of m task positions $\hat{P}_k^i$, $k = 1 \dots m$, for each end-effector (denoted by superscript i), we compute the relative displacements from a selected reference position (let us say, position 1), and equate the relative forward kinematics to those relative positions, for all branches simultaneously,

$$\hat{P}_{1k}^i = \underbrace{\prod_{j=1}^{k_i} e^{\frac{\Delta\hat{\theta}_j^k}{2}\mathsf{S}_j}}_{\text{common}} \underbrace{\prod_{j=k_i+1}^{n_i} e^{\frac{\Delta\hat{\theta}_{i,j}^k}{2}\mathsf{S}_{i,j}}}_{\text{branch}} \qquad \begin{array}{l} i = 1, \dots, b \\ k = 2, \dots, m, \end{array} \tag{2}$$

where the number of common joints is indicated by k_i and the number of end-effectors, or branches, is indicated by b. Using this notation, each branch i has a total of n_i joints, with k_i common joints. The joint axes at the reference configuration are S_j for the common joints and $\mathsf{S}_{i,j}$ for the joints of branch i.

This yields a total of $6(m-1)b$ independent equations to be simultaneously solved.

4 Solvability of Tree Topologies for Exact Synthesis

We define a kinematic chain as solvable if we can find a positive rational number of positions for which the exact dimensional synthesis yields a finite number of solutions. In the case of serial chains the solvability problem is trivial, and we can always find the maximum number of positions for exact synthesis by equating the number of independent unknowns to the number of independent equations, for a chain that does not fully define its group of motion. In the most general case, a serial chain with less than six degrees of freedom is solvable.

When dealing with tree topologies, the task sizing must be done so that the system of equations can be solved simultaneously while not overconstraining any of the branches. The tree topology is solvable if we can find a rational number of positions so that we obtain a finite number of solutions for all branches. In order to do so, some conditions need to be defined. Here we present the theory for the most general case; see [6] for cases restricted to subgroups of the group of rigid motion.

The maximum number of positions for the overall system is computed as follows: let $\mathbf{D}_j^e$ be an $e\times 1$ vector containing the joint degrees-of-freedom for each edge of the contracted graph, and $\mathbf{D}_s^e$ be the $e\times 1$ vector containing the number of structural parameters (four per joint in the general case) for each edge of the contracted graph. Denote as $\mathbf{D}_{ee}^n$ the $b\times 1$ vector containing the degrees-of-freedom of the space of each end-effector, and $\mathbf{D}_c^n$ the $b\times 1$ vector with the number of additionally imposed constraints (if any) for each branch. Define the vectors $\mathbf{B}$ as a $b\times 1$ vector of ones corresponding to branches, or end-effectors, and $\mathbf{E}$ as an $e\times 1$ vector of ones for the edges in the graph considered. The maximum number of positions for the overall graph is given by

$$m = \frac{\mathbf{D}_s^e\cdot\mathbf{E}-\mathbf{D}_c^n\cdot\mathbf{B}}{\mathbf{D}_{ee}^n\cdot\mathbf{B}-\mathbf{D}_j^e\cdot\mathbf{E}}+1. \tag{3}$$

It is necessary that $m\in\mathbb{Q}^+$ for the system to be solvable, but this is not a sufficient condition. In addition, no subgraph starting at the root node and ending at one or more end-effectors can be overdetermined. This phenomenon happens in some topologies with heterogenous branches, such as $2-(1,1,5)$ or $2-(1,5,5)$, to cite a couple of them.

In order to calculate the solvability of each of these subgraphs, use the end-effector path matrix $[\tilde{T}]$ and incidence matrix $[\tilde{B}]$ of the graph [6] to find the vectors $\mathbf{E}_i$ and $\mathbf{B}_i$ containing the edges and branches for a given subgraph i. There are 2^b-2 possible subgraphs for any given rooted tree graph, excluding both the full graph and the null graph; only the non-isomorphic graphs need to be considered.

For each subgraph i, calculate the number of positions needed for exact synthesis of the subgraph,

$$m_i = \frac{\mathbf{D}_s^e \cdot \mathbf{E}_i - \mathbf{D}_c^n \cdot \mathbf{B}_i}{\mathbf{D}_{ee}^n \cdot \mathbf{B}_i - \mathbf{D}_j^e \cdot \mathbf{E}_i} + 1. \tag{4}$$

In addition to this, all different and non-isomorphic subgraphs that appear when exchanging the root node with each of the end-effectors need to be considered. This can be proved using the system of design equations.

Let a tree topology have b branches, the first branching happening after k_1 joints, according to the notation in Eq.(2). For each task position k we can isolate the k_1 common joints by post-multiplying by the inverse forward kinematics corresponding to the rest of joints,

$$\left.\begin{array}{l} \hat{P}_{1k}^1 \left(\prod_{j=k_1+1}^{n_1} e^{\frac{\Delta\hat{\theta}_{1,j}^k}{2}\mathsf{S}_{1,j}} \right)^{-1} = \prod_{j=1}^{k_1} e^{\frac{\Delta\hat{\theta}_j^k}{2}\mathsf{S}_j} \\ \vdots \\ \hat{P}_{1k}^b \left(\prod_{j=k_1+1}^{k_b} e^{\frac{\Delta\hat{\theta}_j^k}{2}\mathsf{S}_j} \prod_{j=k_b+1}^{n_b} e^{\frac{\Delta\hat{\theta}_{b,j}^k}{2}\mathsf{S}_{b,j}} \right)^{-1} = \prod_{j=1}^{k_1} e^{\frac{\Delta\hat{\theta}_j^k}{2}\mathsf{S}_j} \end{array}\right\},$$

and subtract the first equation from the rest, to obtain the new system of $6(m-1)(b-1)$ equations,

$$(\hat{P}_{1k}^1)^{-1}(\hat{P}_{1k}^i) = \left(\prod_{j=k_1+1}^{n_1} e^{\frac{\Delta\hat{\theta}_{1,j}^k}{2}\mathsf{S}_{1,j}} \right)^{-1} \left(\prod_{j=k_1+1}^{k_i} e^{\frac{\Delta\hat{\theta}_j^k}{2}\mathsf{S}_j} \prod_{j=k_i+1}^{n_i} e^{\frac{\Delta\hat{\theta}_{i,j}^k}{2}\mathsf{S}_{i,j}} \right), \quad i = 2\dots b, k = 2\dots m, \tag{5}$$

in which all the unknowns corresponding to the common joints up to the first branching have been eliminated. Also notice that the new task positions correspond to considering the first end-effector as the root node and calculating displacements with respect to a reference frame attached to the new root node. These are the equations for the maximal subgraph not including the previous root node and expressed in this new root node.

We can again repeat the process starting with this new system and eliminating the common joints up to the next branching. At the end of the process, we have explored all non-isomorphic maximal subgraphs created by changing the root node to each of the end-effectors and discarding the previous root node.

As a summary, an overall solution can be imposed only when considering the solvability of all subgraphs that start at the root node and end at end-effectors, including all subgraphs obtained when exchanging the root node with one of the end effectors as described above. In this case, considering m_i as the number of positions for exact synthesis for a subgraph i, with $i \in S$ the set of all possible different end-effector subgraphs up to isomorphism, the topology is solvable if

1. $m \in \mathbb{Q}^+$
2. $m \leq m_i$, $\forall m_i \in \mathbb{Q}^+$, $i \in \mathrm{S}$

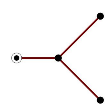
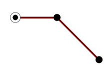
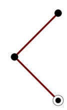
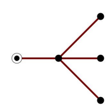
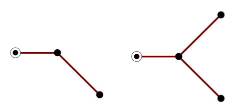
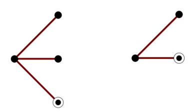
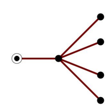
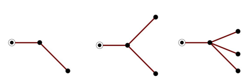
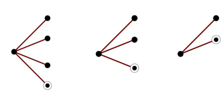
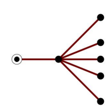
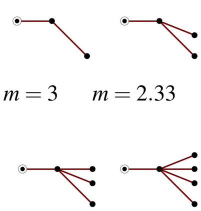
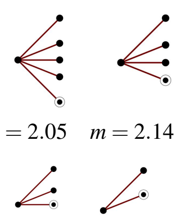
In the case of a subgraph containing c branches and being solvable for $m_i = m$ positions, that subgraph can be solved separately, which eliminates exactly $6c(m-1)$ equations and the same number of unknowns, so that the rest of the graph can be solved a posteriori.

5 Solvable tree topologies with identical branches

Two conditions are established for a topology to be a good candidate for exact synthesis with simultaneous tasks: the topology must be solvable, and it must be solvable for $m \geq 2$ positions.

Considering these criteria, the topologies can be classified according to their solvability. As an example of the classification process, Table 1 presents all possible useful topologies for a single branching with 1-jointed branches consisting of revolute joints, together with the detailed analysis of their solvability.

Table 1 Topologies with 1 common joint and 1-jointed branches

Topology	Graph	Subgraph	Root-change Subgraph	Graph Solvability
$1-(1,1)$	 $m=2.33$	 $m=3$	 $m=3$	Solvable, $m=2.33$
$1-(1,1,1)$	 $m=2.14$	 $m=3$ $m=2.33$	 $m=2.33$ $m=3$	Solvable, $m=2.14$
$1-(1,1,1,1)$	 $m=2.05$	 $m=3$ $m=2.33$ $m=2.14$	 $m=2.14$ $m=2.33$ $m=3$	Solvable, $m=2.05$
$1-(1,1,1,1,1)$	 $m=2$	 $m=3$ $m=2.33$ $m=2.14$ $m=2.05$	 $m=2.05$ $m=2.14$ $m=2.33$ $m=3$	Solvable, $m=2$

Similar analysis can be performed for increasingly complex topologies, however the complete analysis of useful topologies is not possible, due to the fact that many topologies can have as many branches as desired and still get a useful simultaneous task (with 2 or more task positions per finger). Table 2 shows the candidates for dimensional synthesis for the simplest case of branching, a single branching in which all branches have the same number of joints. The minimum and maximum number of branches have been calculated for having an overall task with a finite number of positions, and greater or equal to 2.

Table 2 Maximum and minimum number of branches and solvability for topologies with identical branches consisting of revolute joints $p-(q,\dots,q)$.

Number of joints in wrist (p)	Number of joints in each branch (q)	Number of branches (b)		Solvability
		Minimum	Maximum	(S,NS)
1	1	1	5	S
	2 / 3 / 4	1	∞	S / S / S
	5	2	∞	S
2	1	1	10	S for $b=1,2$
	2 / 3	1	∞	S / S
	4	2	∞	S
	5	3	∞	S
3	1	1	15	S for $b=1$
	2	1	∞	S for $b=1,2,3$
	3 / 4	2	∞	S / S
	5	4	∞	S
4	1	1	20	S for $b=1$
	2 / 3	2	∞	S for $b=2$ / S for $b=2,3,4$
	4	3	∞	S
	5	5	∞	S
5	1	2	25	NS
	2 / 3	2	∞	NS / S for $b=2$
	4	3	∞	S for $b=3,4,5$
	5	6	∞	S

Regarding the solvability of the tree topologies with identical branches, notice that the higher the number of branches, the smaller number of positions m_i obtained: compare the value of Eq.(4) for the overall graph and the graph obtained after eliminating a single branch, and impose non-solvability. Let the tree topology have p common joints and b branches with q joints each, that is, a $p-(q,\dots,q)$ topology with b branches, then the graph is not solvable if

$$\frac{4p+4bq}{6b-p-bq} > \frac{4p+4(b-1)q}{6(b-1)-p-(b-1)q} \quad \Longrightarrow \quad \frac{p}{6-q} < b < \frac{p+6-q}{6-q}. \qquad (6)$$

It can be exhaustively checked for $0<p,q<6$ that there is no positive integer solution for b in this inequality. As a conclusion, all subgraphs starting at the original root node have $m_i \geq m$ when $m_i \in \mathbb{Q}^+$. In order to check for subgraphs when changing the root node, notice that the maximal subgraph obtained is $q - (q, \ldots, q)$ with $b-1$ branches. Following the same reasoning, only this maximal subgraph must be checked and compared to m in each case. The solvability of all useful tree topologies with identical branches presented in the Table 2 has been calculated using this strategy.

6 Conclusions

Dimensional synthesis applied to tree topologies can be used for the design of multi-fingered hands for simultaneous tasks of all fingers. This work focuses on the solvability of tree topologies, that is, on computing which topologies can be synthesized for simultaneous tasks that are meaningful, in which each finger has to reach at least two positions. Previously-stated solvability criteria are proved here and systematically applied to tree topologies with different number of fingers and joints. It turns out that the possible number of fingers is not limited for many topologies, leading to the impossibility of creating a chart of solvable topologies for the general case. It is possible however to classify the topologies with identical fingers, and this classification is presented here. For the general case, a search algorithm is to be developed in future research that can explore solvable topologies for a given task size and for specific requirements on the branches. The results of this research are to be implemented in a general design tool for multi-fingered hands.

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